

16.5 Curl and Divergence

Curl If $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field on \mathbb{R}^3 , and the partial derivatives of P, Q, R exist, the curl of \vec{F} is a vector field on \mathbb{R}^3 defined by

$$\text{curl } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

• Del operator $\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$

• When it operates on a scalar function, it produces the gradient of f :

$$\nabla f = \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

If we think of ∇ as a vector w/ components $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, we can consider the formal cross product of ∇ w/ vector field \vec{F} as follows:

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} \\ &= \text{curl } \vec{F} \end{aligned}$$

To remember $\text{curl } \vec{F} = \nabla \times \vec{F}$

Ex If $\vec{F}(x, y, z) = xyz \hat{i} - x^2 y \hat{k}$, find $\text{curl } \vec{F}$

$$\begin{aligned} \text{curl } \vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 0 & -x^2 y \end{vmatrix} = (-x^2 - 0) \hat{i} - (-2xy - xy) \hat{j} + (0 - xz) \hat{k} \\ &= -x^2 \hat{i} + 3xy \hat{j} - xz \hat{k} \end{aligned}$$

Now given a function of three variables, its gradient is a vector field on \mathbb{R}^3 , and hence we can compute its curl.

Thm If f is a function of three variables that has continuous second order partial derivatives, then $\text{curl}(\nabla f) = 0$.

$$\begin{aligned} \text{Pf } \text{curl}(\nabla f) &= \nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \hat{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \hat{j} \\ &\quad + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \hat{k} \\ &= 0 \quad (\text{by Clairaut's Thm}) \end{aligned}$$

Remember a conservative vector field is one for which $\vec{F} = \nabla f$, and above theorem can be rephrased as :

If \vec{F} is conservative, then $\text{curl}(\vec{F}) = 0$.

Ex Is $\vec{F}(x,y,z) = xyz \hat{i} - x^2y \hat{j} - xz \hat{k}$ conservative ?

- We showed earlier that $\text{curl}(\vec{F}) = -x^2 \hat{i} + 3xy \hat{j} - xz \hat{k} \neq 0$, so \vec{F} is not conservative.

The converse of above thm is not true in general, but it is true if \vec{F} is defined everywhere.

Thm If \vec{F} is a vector field defined on all of \mathbb{R}^3 whose component functions have continuous partial derivatives and $\text{curl} \vec{F} = 0$, then \vec{F} is a conservative field.

Ex Show that $\vec{F}(x,y,z) = yz \hat{i} + xz \hat{j} + (xy + 2z) \hat{k}$ is conservative.

$$\text{Soln } \text{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy + 2z \end{vmatrix} = 0$$

Since $\text{curl } \vec{F} = 0$ and domain of \vec{F} is \mathbb{R}^3 , \vec{F} is a conservative V.F.

Rmk We already showed earlier that $\vec{F} = \nabla f$, where $f(x,y,z) = xyz + z^2$

Divergence

If $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field on \mathbb{R}^3 and $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, $\frac{\partial R}{\partial z}$ exist, then the divergence

of \vec{F} is the function of 3 variable defined by :

$$\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Symbolically $\text{div } \vec{F} = \nabla \cdot \vec{F}$

Rmk

- $\text{curl } \vec{F}$ is a vector field.
- $\text{div } \vec{F}$ is a scalar field.

Ex If $\vec{F} = \langle \ln x, \ln(xy), \ln(xyz) \rangle$, find $\text{div } \vec{F}$.

$$\begin{aligned} \text{Ans } \text{div } \vec{F} &= \nabla \cdot \vec{F} = \frac{\partial}{\partial x} (\ln x) + \frac{\partial}{\partial y} (\ln(xy)) + \frac{\partial}{\partial z} (\ln(xyz)) \\ &= \frac{1}{x} + \frac{x}{xy} + \frac{xy}{xyz} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \end{aligned}$$

Now if \vec{F} is a vector field on \mathbb{R}^3 , then $\text{curl } \vec{F}$ is a vector field, and hence we can compute its divergence.

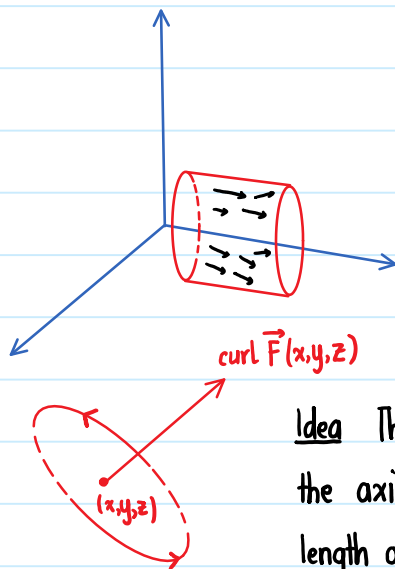
Thm If $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field on \mathbb{R}^3 and P, Q and R have continuous partial derivatives, then $\text{div curl } \vec{F} = 0$.

$$\begin{aligned}
 \text{Pf } \operatorname{div}(\operatorname{curl} \vec{F}) &= \nabla \cdot (\nabla \times \vec{F}) \\
 &= \frac{\partial}{\partial x} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \\
 &= 0 \quad (\text{Clairaut's Thm}).
 \end{aligned}$$

Physical Interpretation of curl and divergence

Ex Imagine a fluid flowing steadily along a pipe and let $\vec{F}(x,y,z)$ be the velocity vector at a point (x,y,z) .

Then \vec{F} assigns to each point (x,y,z) in a domain E , a vector in \mathbb{R}^3 .



- The speed at any point is given by the length of the arrow.

Idea The particle near (x,y,z) in the fluid tend to rotate about the axis that points in the direction of $\vec{\operatorname{curl}} \vec{F}(x,y,z)$ and the length of this curl vector is a measure of how quickly the particle moves around the axis.

- If $\operatorname{curl} \vec{F} = 0$ at a pt P , then fluid is free from rotations and \vec{F} is called irrotational at P .

If \vec{F} is the velocity of a fluid or gas, then $\operatorname{div} \vec{F}(x,y,z)$ represents the net rate of change (wrt time) of the mass of the fluid flowing from the point (x,y,z) per unit volume.

In other words, $\operatorname{div} \vec{F}(x,y,z)$ measures the tendency of the fluid to diverge from the point (x,y,z) .

If $\operatorname{div} \vec{F} = 0$, then \vec{F} is said to be incompressible.

Divergence of gradient $\operatorname{div}(\nabla f) = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$ ($\nabla^2 = \nabla \cdot \nabla$) called the Laplace operator.

Vector Forms of Green's Thm

- Suppose we have a plane region D , its boundary curve C , and functions P and Q satisfy hypothesis of Green's Thm.

Let $\vec{F} = P\hat{i} + Q\hat{j}$. Then,

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P dx + Q dy \quad \text{and regarding } \vec{F} \text{ as a vector field on } \mathbb{R}^3 \text{ w/ third component } 0,$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

and,

$$(\text{curl } \vec{F}) \cdot \hat{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} \cdot \hat{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and therefore,

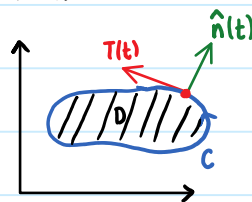
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl } \vec{F}) \cdot \hat{k} dA.$$

Here we express the line integral of the tangential component of \vec{F} along C as the double integral of the vertical component of $\text{curl } \vec{F}$ over the region D enclosed by C .

- Now we will derive similar formula involving normal component of \vec{F} .

If C is given by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, $a \leq t \leq b$, then

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \left\langle \frac{x'(t)}{|\vec{r}'(t)|}, \frac{y'(t)}{|\vec{r}'(t)|} \right\rangle$$



Then the outward unit normal vector to C is

given by

$$\hat{n}(t) = \left\langle \frac{y'(t)}{|\vec{r}'(t)|}, -\frac{x'(t)}{|\vec{r}'(t)|} \right\rangle \quad \left[\text{check } T(t) \cdot \hat{n}(t) = 0 \text{ and } |\hat{n}(t)| = 1 \right]$$

Then,

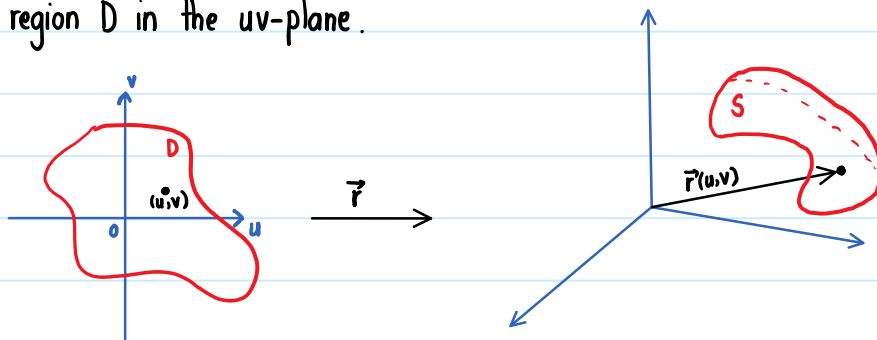
$$\begin{aligned}
 \oint_C \vec{F} \cdot \hat{n} \, ds &= \int_a^b (\vec{F} \cdot \hat{n})(t) |\vec{r}'(t)| \, dt \\
 &= \int_a^b \left[P(x(t), y(t)) \frac{y'(t)}{|\vec{r}'(t)|} - Q(x(t), y(t)) \frac{x'(t)}{|\vec{r}'(t)|} \right] |\vec{r}'(t)| \, dt \\
 &= \int_a^b [P(x(t), y(t)) y'(t) \, dt - Q(x(t), y(t)) x'(t) \, dt] \\
 &= \int_C P \, dy - Q \, dx \stackrel{G.T.}{=} \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_D \operatorname{div} \vec{F}(x, y) \, dA
 \end{aligned}$$

The line integral of the normal component of \vec{F} along C is equal to the double integral of the divergence of \vec{F} over the region D enclosed by C .

16.6 Parametric Surfaces

In the same way we described a space curve by a vector function $\vec{r}(t)$ of a single parameter t , we can describe a surface by a vector function $\vec{r}(u, v)$ of two parameters u and v .

So, $\vec{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$ is a vector valued function defined on a region D in the uv -plane.



The surface S consists of all points $(x, y, z) \in \mathbb{R}^3$ such that $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ where (u, v) varies throughout D , and is called a parametric surface S given by parametric equations.

Ex Identify and sketch the surface w/ vector equation

$$\vec{r}(u,v) = 2\cos u \hat{i} + 2\sin u \hat{j} + v\hat{k}$$

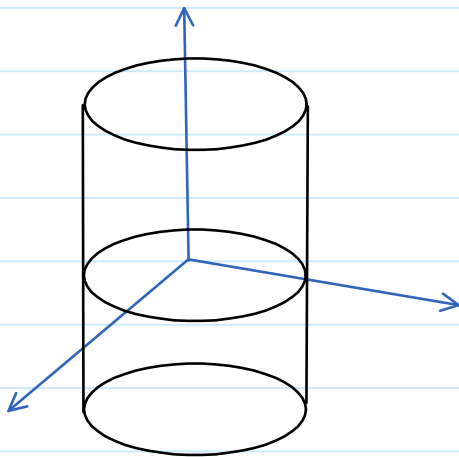
Soln The parametric equation of the surface are

$$x(u,v) = 2\cos u, \quad y(u,v) = 2\sin u, \quad z(u,v) = v$$

For any point (x,y,z) on the surface S , we have

$x^2 + y^2 = 4$ meaning all cross-sections keeping z -constant i.e parallel to the xy -plane are circles of radius 2.

Now $z = v$ and as no restrictions are placed on v , the surface is a circular cylinder w radius 2, whose axis is the z -axis.

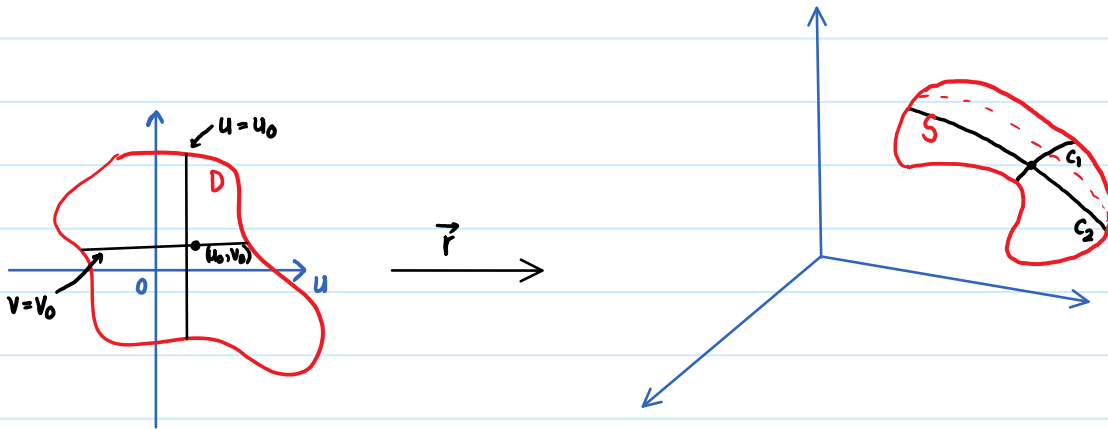


So here we placed no restrictions on the parameters u and v and therefore we get an entire cylinder.

But let's restrict u and v (the parameter domains as $0 \leq u \leq \pi/2$ and $0 \leq v \leq 1$).

- Then $x \geq 0, y \geq 0, 0 \leq z \leq 1$ and we get the quarter cylinder of length 1.

For a parametric surface S given by $\vec{r}(u, v)$, there are two families of curves that we want to keep track of, one family w/ u constant and the other w/ v constant. These are vertical and horizontal lines in the uv -plane.



If we set u to be constant i.e. $u = u_0$, then $\vec{r}(u_0, v)$ becomes a vector function of a single parameter v and defines a curve C_1 lying on S .

Similarly, if we keep v constant by setting $v = v_0$, we get a curve C_2 given by $\vec{r}(u, v_0)$ lying on S . These are called grid curves.

We are going to be interested in finding a vector function to represent a given surface.

Ex The vector function that represents the plane that passes through the point P_0 w/ position vector \vec{r}_0 and that contains two non parallel vectors \vec{a} and \vec{b} is given by $\vec{r}(u, v) = \vec{r}_0 + u\vec{a} + v\vec{b}$, where u and v are real numbers.

If we write $\vec{r} = \langle x, y, z \rangle$, $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then

- $x = x_0 + ua_1 + vb_1$
- $y = y_0 + ua_2 + vb_2$
- $z = z_0 + ua_3 + vb_3$

Ex Find a parametric equation for the sphere $x^2 + y^2 + z^2 = a^2$.

Ans In spherical coordinates, $\rho = a$, so we can choose ϕ, θ in spherical coordinates as parameters.

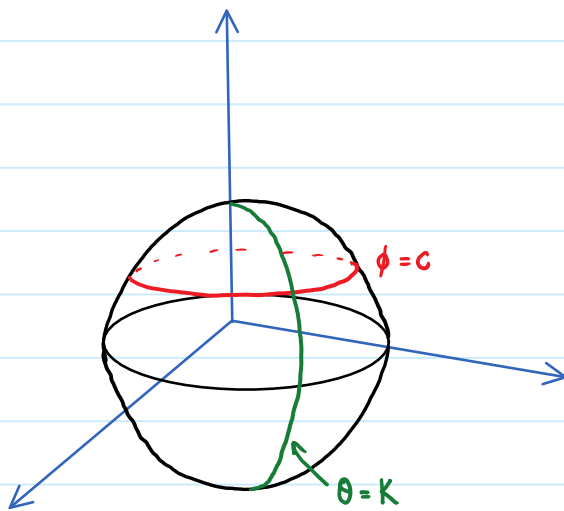
Then putting $\rho = a$ in the equation for conversion from spherical to rectangular coordinates and we get

$$x = a \sin \phi \cos \theta, \quad y = a \sin \phi \sin \theta, \quad z = a \cos \phi, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi.$$

i.e. our parameter domain is the rectangle $D = [0, \pi] \times [0, 2\pi]$.

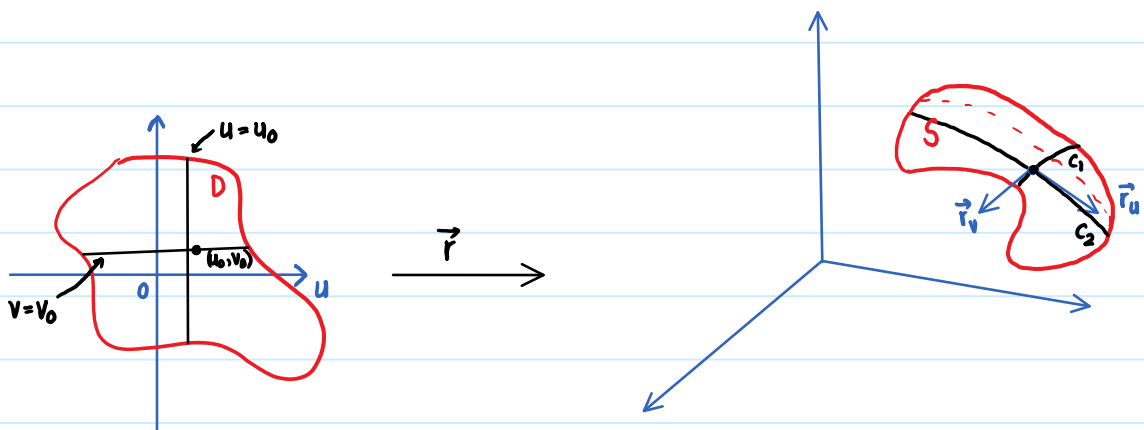
The grid curves with $\phi = c$ are the circles of constant latitude

The grid curves with $\theta = c$ are the meridians (semicircles) that connect the north and south poles.



Tangent planes

S be a surface given by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ at a position vector $\vec{r}(u_0, v_0)$.



The tangent vector to C_1 ($\vec{r}(u_0, v)$) is obtained by taking partial derivatives of \vec{r} wrt v

$$\vec{r}_v = \frac{\partial x}{\partial v}(u_0, v_0) \hat{i} + \frac{\partial y}{\partial v}(u_0, v_0) \hat{j} + \frac{\partial z}{\partial v}(u_0, v_0) \hat{k}$$

The tangent vector to C_2 ($\vec{r}(u, v_0)$) is obtained by taking partial derivatives of \vec{r} wrt u

$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0, v_0) \hat{i} + \frac{\partial y}{\partial u}(u_0, v_0) \hat{j} + \frac{\partial z}{\partial u}(u_0, v_0) \hat{k}$$

If $\vec{r}_u \times \vec{r}_v$ is not $\vec{0}$, then the surface S is called smooth (it has no "corners").

Then for a smooth surface, the tangent plane is the plane that contains the tangent vector \vec{r}_u and \vec{r}_v and the vector $\vec{r}_u \times \vec{r}_v$ is normal vector to the tangent plane.

Ex Find the tangent plane to the surface w/ parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $(1, 1, 3)$.

Soln First compute the tangent vectors :

$$\vec{r}_u = \frac{\partial x}{\partial u} \hat{i} + \frac{\partial y}{\partial u} \hat{j} + \frac{\partial z}{\partial u} \hat{k} = 2u \hat{i} + \hat{k}$$

$$\vec{r}_v = \frac{\partial x}{\partial v} \hat{i} + \frac{\partial y}{\partial v} \hat{j} + \frac{\partial z}{\partial v} \hat{k} = 2v \hat{j} + 2 \hat{k}.$$

Then the normal vector to the tangent plane is

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v \hat{i} - 4u \hat{j} + 4uv \hat{k}$$

- Now we need to find the parameter values corresponding to the point $(1, 1, 3)$.
 $u^2 = 1$, $v^2 = 1$, $3 = u + 2v \Rightarrow u = 1$, $v = 1$.

So, $\vec{r}_u \times \vec{r}_v (1, 1) = -2\hat{i} - 4\hat{j} + 4\hat{k}$

Then the equation of the tangent plane at $(1, 1, 3)$ is

$$\begin{aligned} -2(x-1) - 4(y-1) + 4(z-3) &= 0 \\ \Rightarrow x + 2y - 2z + 3 &= 0. \end{aligned}$$